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Discrete q-derivatives and symmetries of q-difference equations

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Abstract
In this paper we extend the *umbral calculus*, developed to deal with difference equations on uniform lattices, to q-difference equations. We show that many properties considered for shift invariant difference operators satisfying the umbral calculus can be implemented to the case of the q-difference operators. This q-*umbral calculus* can be used to provide solutions to linear q-difference equations and q-differential delay equations. To illustrate the method, we will apply the obtained results to the construction of symmetry solutions for the q-heat equation.

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1. Introduction

Functions depending on q-variables appear in many physical problems. They enter in the study of exactly solvable models in statistical mechanics [1], in conformal field theory [2] and are thus very relevant for applications. For example, q-exponential distributions can be obtained following Gibbs' procedure from the stationary conditions on a certain generalized entropy [3]. Standard q-exponential functions are also used to extrapolate between the Fermi–Dirac ($q = \infty$) and Bose–Einstein ($q = 0$) statistics, passing through the Maxwell–Boltzmann ($q = 1$) statistics [4].

In the case of difference equations, one had proved [5] that there exists a very powerful method for systematically discretizing linear differential equations while preserving their properties. Here we extend that method to the case of q-difference equations. We can show that many properties considered in [5] for shift invariant difference operators satisfying the umbral calculus [6–8] can be extended to the case of the q-difference operators considered in [9–11]. For any q-difference operator this q-umbral calculus can be applied to provide solutions to linear q-difference equations and q-differential delay equations. As an illustration, we will apply the method in the construction of symmetry solutions for the q-heat equation.

In section 2 we will define a q-difference equation in p independent variables and, for the sake of simplicity, just one dependent variable and characterize the symmetry transformations which will leave the equation invariant. Section 3 is devoted to the study of the properties of q-calculus, showing the differences and the similarities between differential and q-difference calculus. In particular, in section 4 we will discuss from an analytic and numerical point of view the simplest q-functions. In section 5, we present the symmetries of a q-heat equation using the correspondence between q-calculus and differential calculus. Section 6 is dedicated to a few conclusive remarks.

2. q-Difference equations and its Lie symmetries

Let us consider a linear q-difference equation, involving, for notational simplicity, only one scalar function $u(x)$ of p independent variables $x = (x_1, x_2, \ldots, x_p)$ evaluated at a finite number of points of a nonuniform lattice characterized by positive parameters $q = (q_1, q_2, \ldots, q_p)$. Symbolically we write

$$E_N\big(x, T^a u(x), T^{a_{i_1}} \Delta_{x_{i_1}} u(x), T^{a_{i_1 i_2}} \Delta_{x_{i_1}} \Delta_{x_{i_2}} u(x), \ldots, T^{a_{i_1 i_2 \ldots i_N}} \Delta_{x_{i_1}} \Delta_{x_{i_2}} \ldots \Delta_{x_{i_N}} u(x)\big) = 0$$
$$a = (a_1, a_2, \ldots, a_p) \tag{2.1}$$

where E_N is some given function of its arguments, $a, a_{i-1}, a_{i_1 i_2}, \ldots$ are multiindices and $i_1, i_2, \ldots, i_N$ take values between 1 and p. By $T^a u(x)$ we mean

$$T^a u(x) = T_{x_1}^{a_1} T_{x_2}^{a_2} \ldots T_{x_p}^{a_p} u(x)$$

where a_i, with $i = 1, 2, \ldots, p$, takes values between m_i and n_i, with m_i, n_i being fixed integers ($m_i \leqslant n_i$), and the individual q-shift operator is given by

$$T_{x_i}^{a_i} u(x) = u\big(x_1, x_2, \ldots, x_{i-1}, q_i^{a_i} x_i, x_{i+1}, \ldots, x_p\big). \tag{2.2}$$

The other q-shift operators $T^{a_{i_1 i_2}}, \ldots$ are defined in a similar way. The operator Δ_{x_i} is a q_i-difference operator such that when $q_i \to 1$ goes into the partial derivative with respect to the x_i variable.

To study the symmetries of equation (2.1) we will use the approach introduced in [12], based on the formalism of evolutionary vector fields for differential equations [13]. As in the case of differential equations, the symmetry group of a discrete equation is characterized by those transformations of the equation that carry solutions $u(x)$ into solutions $\tilde{u}(x)$. Moreover, we look only for those symmetries which in the continuous limit go over to Lie point symmetries. In such a case the infinitesimal symmetry generators of the symmetry group of equation (2.1) in evolutionary form have the general expression

$$X_e \equiv Q(x, u)\partial_u = \left(\sum_{i=1}^{p} \xi_i\big(x, T^a u, \{q_j\}_{j=1}^p\big) T^b \Delta_{x_i} u - \phi\big(x, T^c u, \{q_j\}_{j=1}^p\big)\right) \partial_u \tag{2.3}$$

with $\xi_i\big(x, T^a u, \{q_j\}_{j=1}^p\big)$ and $\phi\big(x, T^c u, \{q_j\}_{j=1}^p\big)$ such that in the continuous limit go over $\xi_i(x, u)$ and $\phi(x, u)$, respectively (the infinitesimal generators of the corresponding Lie point symmetries).

As equation (2.1) is of order N in the difference operators, the N-th prolongation of X_e will verify the invariance condition

$$pr^N X_e E_N|_{E_N=0} = 0. \tag{2.4}$$

Note that the expressions involved in (2.3), (2.4) are analogous to those of the continuous case [13] and can be derived in a similar way (for more details see [12]).

The symmetries of equation (2.1) are given by condition (2.4), which give rise to a set of determining equations for ξ_i and ϕ.

The formalism presented above may become quite involved. The situation is simpler for linear equations where we can use a reduced ansatz. In this case, we can assume that the evolutionary vectors (2.3) have the form

$$X_e = \left(\sum_i \xi_i(x, T^a, q_j)\Delta_{x_i} u - \phi(x, T^a, q_j) u \right) \partial_u. \tag{2.5}$$

The vector fields, X_e, can be written as $X_e = (\hat{X}u)\partial_u$ with

$$\hat{X} = \sum_i \xi_i(x, T^a, q_j)\Delta_{x_i} - \phi(x, T^a, q_j). \tag{2.6}$$

However, in general, the resulting symmetries span only a subalgebra of the whole Lie symmetry algebra (see [12]). If the system is nonlinear the simplification (2.5) is too restrictive and is almost impossible to get a non-trivial result as, *a priori*, we need to consider an infinite number of terms.

3. q-Calculus

In this section we will present the generalities of q-calculus [14]. We will restrict ourselves for the sake of simplicity to one independent variable. Moreover, in the following we will consider just the simplest q-derivatives (at the right, at the left and symmetric, respectively)

$$\Delta_x^+ = \frac{1}{q_x^+ x}(T_x - 1) \qquad \Delta_x^- = \frac{1}{q_x^- x}\left(1 - T_x^{-1}\right) \qquad \Delta_x^{\mathrm{s}} = \frac{1}{q_x^{\mathrm{s}} x}\left(T_x - T_x^{-1}\right) \tag{3.1}$$

where q_x is a real dilation positive parameter associated with the variable x and q_x^i, $i = \pm, s$, are given by

$$q_x^+ = q_x - 1 \qquad q_x^- = 1 - \frac{1}{q_x} \qquad q_x^{\mathrm{s}} = q_x - \frac{1}{q_x}.$$

The operator T_x is a q-dilation operator

$$T_x f(x) = f(q_x x) \qquad T_x^{-1} f(x) = f(x/q_x) \tag{3.2}$$

the one-dimensional reduction of the one defined in equation (2.2). Formally we have

$$T_x = q_x^{x\partial_x} = \mathrm{e}^{\log q_x x \partial_x}. \tag{3.3}$$

When we do not specify which q-derivative we are using we will write just Δ_x.

It is easy to see that, due to the form of the q-derivative considered (3.1), the shift operator and the q-derivatives do not commute. So, they do not satisfy one of the basic conditions in the umbral calculus [6, 7]. We can, however, define the *q-umbral calculus* in a way similar to the standard umbral calculus avoiding the above requirement.

We can easily find that

$$[\Delta_x, x] = \begin{cases} T_x & \text{for} \quad \Delta_x^+ \\ T_x^{-1} & \text{for} \quad \Delta_x^- \\ \dfrac{1}{1+q_x}\left(q_x T_x + T_x^{-1}\right) & \text{for} \quad \Delta_x^{\mathrm{s}}. \end{cases} \tag{3.4}$$

If instead of the standard commutator, we consider the q-commutators defined as $[A, B]_{q^+} = AB - qAB$ and $[A, B]_{q^-} = AB - (1/q)AB$, we have

$$[\Delta_x, x]_q = 1. \tag{3.5}$$

This result cannot be extended to the case of the symmetric q-derivative Δ_x^s. Moreover, since the expression (3.5) does not satisfy the Leibniz rule, the determining equations (2.4) become very complicate. So, in the following we will consider just standard commutators.

In the spirit of umbral calculus [7, 8], we can define an operator β_x, depending on T_x, such that

$$[\beta_x, T_x] = 0 \qquad [\Delta_x, \beta_x x] = 1.$$

For the three q-derivatives introduced above (3.1) we can find the following explicit expressions of β_x:

$$\begin{aligned} \beta_x^+ &= (q_x - 1)x\partial_x(T_x - 1)^{-1} = q_x^+ x\partial_x(T_x - 1)^{-1} \\ \beta_x^- &= \left(1 - \frac{1}{q_x}\right)x\partial_x\left(1 - T_x^{-1}\right)^{-1} = q_x^- x\partial_x\left(1 - T_x^{-1}\right)^{-1} \\ \beta_x^{\mathrm{s}} &= \left(q_x - \frac{1}{q_x}\right)x\partial_x\left(T_x - T_x^{-1}\right)^{-1} = q_x^{\mathrm{s}} x\partial_x\left(T_x - T_x^{-1}\right)^{-1}. \end{aligned} \tag{3.6}$$

It is easy to prove that in all the above cases $\beta_x x \Delta_x = x\partial_x$. However, this may not be the only possible definition of β_x. Let us note that, due to the presence of the ∂_x operator in the definition of β_x, the q-umbral correspondence of an explicitly x-dependent differential equation will give rise to a q-differential delay equation [15].

We can reexpress the functions β_x as an infinite series in terms of the shift operators, thus proving that they commute with the shift operators. From (3.3) we get

$$x\partial_x = \frac{1}{\log q_x}\ln T_x = \frac{1}{\log q_x}\ln(1 + (T_x - 1)) = \frac{1}{\log q_x}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(T_x - 1)^n}{n}.$$

Consequently

$$\beta_x^+ = \frac{q_x^+}{\log q_x^+}\sum_{n=0}^{\infty}(-1)^n\frac{(T_x - 1)^n}{n+1}.$$

Similarly from

$$x\partial_x = -\frac{1}{\log q_x}\ln T_x^{-1} = -\frac{1}{q_x - 1}\ln\left(1 + \left(T_x^{-1} - 1\right)\right) = \frac{1}{\log q_x}\sum_{n=1}^{\infty}(-1)^n\frac{\left(T_x^{-1} - 1\right)^n}{n}$$

we get

$$\beta_x^- = \frac{q_x^-}{\log q_x^+}\sum_{n=0}^{\infty}(-1)^n\frac{\left(T_x^{-1} - 1\right)^n}{n+1} = \frac{q_x^-}{\log q_x^+}\sum_{n=0}^{\infty}\frac{\left(1 - T_x^{-1}\right)^n}{n+1}.$$

Finally, for the symmetric derivative, as

$$\frac{T_x - T_x^{-1}}{2} = \sinh(\log q_x x\partial_x)$$

we find that

$$x\partial_x = \frac{1}{\log q_x}\sinh^{-1}\frac{T_x - T_x^{-1}}{2} = \frac{1}{\log q_x}\sum_{n=1}^{\infty}(-1)^{n+1}C_n\frac{\left(T_x - T_x^{-1}\right)^{2n-1}}{2^{2n-1}}$$

where the coefficients C_n are given by

$$C_1 = 1 \qquad C_n = \frac{\prod_{k=2}^{n}(2k-3)}{(2n-1)\prod_{k=2}^{n}(2k-2)} \qquad \forall n \geqslant 2.$$

So, we have

$$\beta_x^{\mathrm{s}} = \frac{q_x^{\mathrm{s}}}{\log q_x} \sum_{n=0}^{\infty} (-1)^n C_{n+1} \frac{\left(T_x - T_x^{-1}\right)^{2n}}{2^{2n+1}}.$$

For functions f and g, entire in $\beta_x x$, the Leibniz rule takes the form

$$[\Delta_x, fg] = [\Delta_x, f]g + f[\Delta_x, g]. \tag{3.7}$$

Note that the formal expression (3.7) of Leibniz's rule is the same as in the case of discrete derivatives [5]. For the sake of brevity we will write $D_x f := [\Delta_x, f]$. The expression (3.7) is operatorial, if we want to have a functional expression we need to project it by acting on a constant function as 1. In this case, however, the Leibniz rule (3.7) is a trivial identity.

Taking into account Leibniz's rule (3.7) we can prove the following property

$$[\Delta_x, (\beta_x x)^n] = D_x(\beta_x x)^n = n(\beta_x x)^{n-1} \qquad \forall n \in \mathbb{N} \tag{3.8}$$

thus showing that $(\beta_x x)^n$ are basic polynomials for the operator D_x, and, when projected, for Δ_x. So, we have defined an operator D_x which on functions of $\beta_x x$ have the same properties as the normal derivatives ∂_x on functions of x. Therefore, we can say, roughly speaking, that whatsoever is valid for differential equations may also be valid for the D_x operators, provided we substitute formally in the corresponding entire functions the variable x by $\beta_x x$. This general idea is the content of the q-*umbral correspondence*. In the case of linear equations, when the derivations act linearly on functions, the projection procedure will transform the operator D_x into Δ_x and we will get q-difference equations.

Let us analyse the meaning of the operators $(\beta_x x)^n$ for the three q-derivative operators. Since

$$(T_x - 1)x = x(q_x T_x - 1) \qquad (T_x - 1)^{-1}x = x(q_x T_x - 1)^{-1}$$

$$\left(1 - T_x^{-1}\right)x = x\left(1 - \frac{1}{q_x}T_x^{-1}\right) \qquad \left(1 - T_x^{-1}\right)^{-1}x = x\left(1 - \frac{1}{q_x}T_x^{-1}\right)^{-1}$$

$$\left(T_x - T_x^{-1}\right)x = x\left(q_x T_x - \frac{1}{q_x}T_x^{-1}1\right) \qquad \left(T_x - T_x^{-1}\right)^{-1}x = x\left(q_x T_x - \frac{1}{q_x}T_x^{-1}\right)^{-1}$$

we get

$$\begin{aligned}
\beta_x^+ x &= q_x^+ x(1 + x\partial_x)(q_x T_x - 1)^{-1} \\
\beta_x^- x &= q_x^- x(1 + x\partial_x)\left(1 - \frac{1}{q_x}T_x^{-1}\right)^{-1} \\
\beta_x^{\mathrm{s}} x &= q_x^{\mathrm{s}} x(1 + x\partial_x)\left(q_x T_x - \frac{1}{q_x}T_x^{-1}\right)^{-1}.
\end{aligned} \tag{3.9}$$

Let us stress once more that these expressions have an operator character. In order to have a functional nature we have to project them by acting on a constant. So, taking into account that $T_x 1 = 1$ and $T_x^{-1} 1 = 1$, we have

$$\beta_x x 1 = x. \tag{3.10}$$

Starting from (3.10) we can demonstrate by induction that

$$(\beta_x x)^n 1 = (\beta_x x)(\beta_x x)\ldots(\beta_x x)1 = \frac{n!}{[n]_q!}x^n \qquad \forall n \in \mathbb{N}^+$$

where

$$[n]_q^+ = \frac{q^n - 1}{q - 1} \qquad [n]_q^- = \frac{1 - q^{-n}}{1 - q^{-1}} \qquad [n]_q^s = \frac{q}{q^2 - 1}\frac{q^{2n} - 1}{q^n} = \frac{1}{q^{n-1}}\sum_{k=1}^{n} q^{2k-2}$$

and

$$[n]_q! = [n]_q[n-1]_q \dots [1]_q.$$

Consequently, if we consider an entire function such as the exponential, we have

$$\mathrm{e}^{\lambda\beta_x x}1 = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(\beta_x x)^n 1 = \sum_{n=0}^{\infty} \lambda^n \frac{x^n}{[n]_q!}.$$

So, by the q-umbral correspondence, the exponential function becomes

$$\mathrm{e}^{\lambda x} = \sum_{n=0}^{\infty} \lambda^n \frac{x^n}{n!} \longrightarrow \mathrm{e}^{\lambda\beta_x x}1 = \sum_{n=0}^{\infty} \lambda^n \frac{x^n}{[n]_q!}. \tag{3.11}$$

Let us consider the Gaussian function that we will be using later on. It takes the form

$$\mathrm{e}^{-\lambda(\beta_x x)^2}1 = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}(\beta_x x)^{2n}1 = \sum_{n=0}^{\infty} (-\lambda)^n \frac{(2n)!}{n!} \frac{x^{2n}}{[2n]_q!}. \tag{3.12}$$

Moreover, for an arbitrary point x_0 and an arbitrary constant a, we can prove that

$$D_x(\beta_x x + x_0)^a = a(\beta_x x + x_0)^{a-1} \qquad \forall a \in \mathbb{R}.$$

The proof is based on the idea that the differential equation

$$(x + x_0)\partial_x f = af$$

whose solution is $f = (x + x_0)^a$ can be transformed by the q-umbral correspondence into the discrete equation

$$(\beta_x x + x_0)D_x(\beta_x x + x_0)^a = a(\beta_x x + x_0)^a$$

which has formally the same solution in power series (substituting the x of the solution of the continuous equation by $\beta_x x$ in the discrete case). Indeed the expression

$$(x + x_0)^a = \sum_{n=0}^{\infty} \frac{x_0^{a-n}\prod_{k=0}^{n-1}(a-k)}{n!} x^n$$

is replaced by

$$(\beta_x x + x_0)^a = \sum_{n=0}^{\infty} \frac{x_0^{a-n}\prod_{k=0}^{n-1}(a-k)}{n!} (\beta_x x)^n$$

and, after projection, we get

$$(\beta_x x + x_0)^a 1 = \sum_{n=0}^{\infty} \frac{x_0^{a-n}\prod_{k=0}^{n-1}(a-k)}{n!} (\beta_x x)^n 1 = \sum_{n=0}^{\infty} \frac{x_0^{a-n}\prod_{k=0}^{n-1}(a-k)}{[n]_q!} x^n.$$

4. q-Umbral functions

In this section we will examine in detail some basic discrete functions obtained by the q-umbral method. There are some points to be investigated: (i) as the q-umbral correspondence applies to series expansions, it is important to know the radius of convergence for the resulting series; (ii) it is of interest in physical applications to keep track of the modifications that the q-umbral calculus introduce in the behaviour of the continuous functions, in particular on their asymptotic behaviour.

In the following examples we shall consider as q-derivative operator the right Δ^+ and the symmetric Δ^s q-derivative. We will analyse here two of the functions which appear in the symmetry reduction of the heat equation, the exponential (3.11) and Gaussian (3.12) functions, which exemplify the kind of results one can obtain from the q-umbral calculus for entire functions. These functions are described by the ordinary differential equations

$$\frac{\mathrm{d}f_e(x)}{\mathrm{d}x} = \lambda f_e(x) \tag{4.1}$$

and

$$\frac{\mathrm{d}f_g(x)}{\mathrm{d}x} = -2\lambda x f_g(x) \tag{4.2}$$

whose solutions, up to constant factors, are given by $f_e(x) = \mathrm{e}^{\lambda x}$ and $f_g(x) = \mathrm{e}^{-\lambda x^2}$, respectively. We will deal with the q-umbral analogues of equations (4.1), (4.2) and look for their discrete solutions.

4.1. q-Exponential functions

From the q-umbral correspondence the difference equation satisfied by the q-exponential $E_q(x)$ is (see equation (4.1))

$$\Delta E_q(x) = \lambda E_q(x). \tag{4.3}$$

We will consider always the domain $x > 0$. The negative values of x can be obtained by changing the sign of λ.

4.1.1. The right exponential. The difference equation (4.3) becomes

$$E_q(qx) = [1 + (q-1)\lambda x]E_q(x). \tag{4.4}$$

The solution can be expressed as a product

$$E_q(q^n x_0) = \prod_{j=0}^{n-1}[1 + (q-1)\lambda q^j x_0]E_q(x_0) \qquad n \in \mathbb{N}^+ \tag{4.5}$$

$$E_q(q^{-n} x_0) = \prod_{j=1}^{n}\frac{1}{[1 + (q-1)\lambda q^{-j} x_0]}E_q(x_0) \qquad n \in \mathbb{N}^+. \tag{4.6}$$

We have four different types of behaviour of the q-exponential function according to the values of q and λ, namely $q \gtrless 1$ and $\lambda \lesseqgtr 0$.

- $q > 1$
  - (i) Let us at first consider the case $\lambda > 0$. The recurrence of the difference equation implies that $E_q(x)$ is a monotonously increasing function of x.

    By the q-umbral correspondence the solution of the equation (4.1) can also be obtained by q-umbralizing the series representation of the exponential function. In such a case we have

    $$\widetilde{E}_q(x) = \sum_{k=0}^{\infty}\frac{(\lambda x)^k (q-1)^k}{\prod_{j=1}^{k}(q^j - 1)}. \tag{4.7}$$

    This solution converges for all $x > 0$ so that it gives the unique (up to a constant) solution (4.5), (4.6) of (4.4) in all the domain.

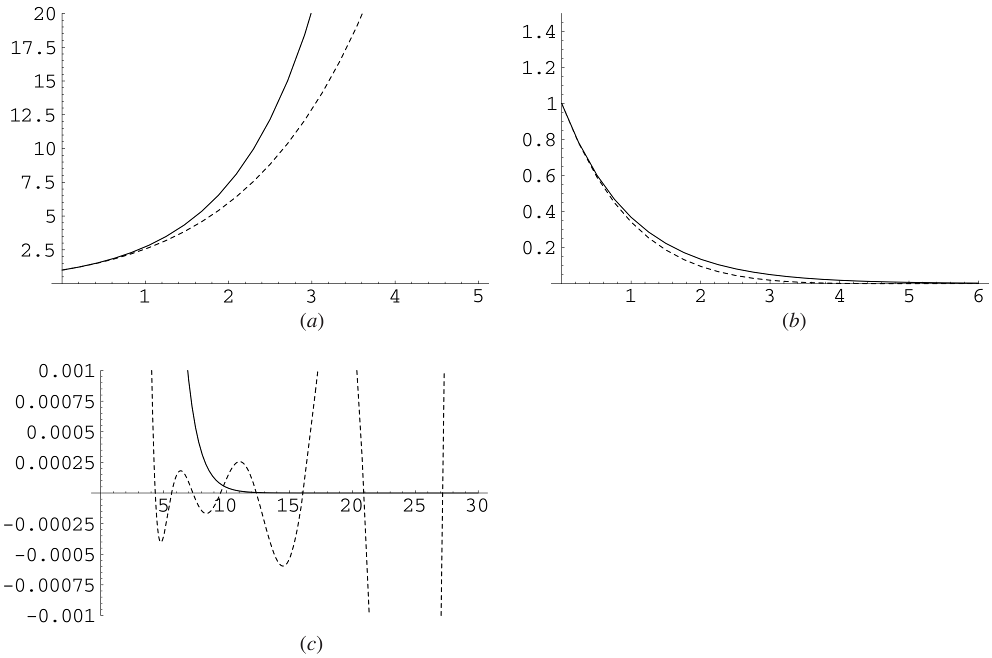


Figure 1. (*a*) Plot of the exponential function (continuous line) and right q-exponential function (dashed line) for $\lambda = 1$ and $q = 1.3$. (*b*) Plot of the exponential function (continuous line) and right q-exponential function (dashed line) for $\lambda = -1$ and $q = 1.3$. (*c*) Enlarged plot of the exponential function (continuous line) and right q-exponential function (dashed line) for $\lambda = -1$ and $q = 1.3$.

(ii) In the case $\lambda < 0$ we see that the recurrence leads to a decreasing function as long as $1 - (q-1)\lambda x > 0$. However, for further values of x, where $1 - (q-1)\lambda x < 0$ the function oscillates with higher diverging amplitudes. For the particular point $x_0 = 1/(q-1)|\lambda|$ we have $E_q(qx_0) = 0$, therefore also $E_q(q^n x_0) = 0, \forall n \in \mathbb{N}$. This means that in the q-lattice $x_n = x_0 q^n, n \in \mathbb{Z}$, the q-exponential decreases for n negative, i.e. for values of x less than x_0 and vanishes after x_0, for n positive, avoiding the oscillations. As $q \to 1$ the point $x_0 \to \infty$, so that the q-exponential becomes closer and closer to the (continuous) exponential. The q-umbral function (4.7) displays these features as it is shown in figure 1.

- $q < 1$

(i) For $\lambda > 0$ and for large values of n one can always find $[1+(q-1)\lambda q^n x_0] > 0$, and the solution is a monotonous increasing function. For x_0 such that $[1+(q-1)\lambda x_0] = 0$, the solution is not defined. In fact in that point it diverges and, consequently, also diverges in the lattice points $x_0 q^{-n}, n \in \mathbb{N}$. For the points $x_j = x_0 q^j$ such that $[1+(q-1)\lambda q^j x_0] < 0$, the solution oscillates, changing its sign alternatively, with divergences at the separating points. The amplitude of the oscillations tends to zero, as $x \to +\infty$.

(ii) For $\lambda < 0$ the solution is a monotonously decreasing function which, in the limit $x \to 0$, tends to zero.

The radius of convergence of the q-umbral series is given by $R = 1/|q-1||\lambda|$. The q-umbral function $\widetilde{E}_q(x)$ has two vertical asymptotes at the symmetric points

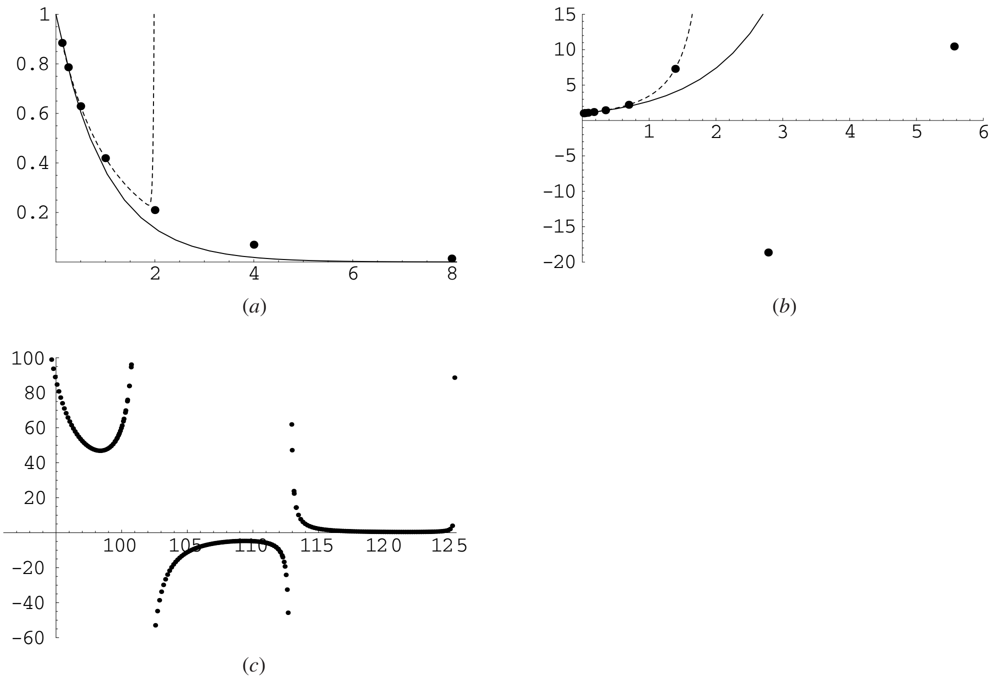


Figure 2. (a) Plot of the exponential function (continuous line) and right q-exponential function (dashed line) and the solution of equation (4.4) given by the points, for $\lambda = -1$ and $q = 0.5$. (b) Plot of the exponential function (continuous line) and right q-exponential function (dashed line) and the solution of equation (4.4) given by the points, for $\lambda = 1$ and $q = 0.5$. (c) Enlarged plot of the solution of equation (4.4), for $\lambda = 1$ and $q = 0.5$.

$x = \pm R$. Therefore, inside the range of convergence (where there are no oscillations) the q-umbral function reproduces correctly the solution $E_q(x)$, but for $x > |R|$, where it does not converge, it supplies no information about the solution. These features are illustrated in figure 2.

4.1.2. The symmetric exponential. In this case the difference equation (4.3) becomes a three-term relation given by

$$E_q^s(qx) = (q - 1/q)\lambda x E_q^s(x) + E_q^s(x/q). \tag{4.8}$$

There are two independent solutions for the symmetric q-exponential. It is not necessary to distinguish the cases $q > 1$ and $q < 1$ because they play a symmetric role. For $\lambda > 0$ the relation (4.8) gives growing functions in the x variable. For $\lambda < 0$ the function initially is decreasing but after a certain point the discrete solutions start to oscillate.

The q-umbral solution to the recurrence (4.8) is given by

$$\widetilde{E}_q^s(qx) = \sum_{k=0}^{\infty} \frac{(\lambda x)^k (q - 1/q)^k}{\prod_{j=1}^{k}(q^j - 1/q^j)}. \tag{4.9}$$

This series converges for all x, so that it provides one of the solutions to the recurrence (4.8), and in the limit $q \to 1$ it goes into the continuous exponential. The behaviour of the umbral $\widetilde{E}_q^s$ presents the features of the recurrence above described, i.e. it approaches the continuous

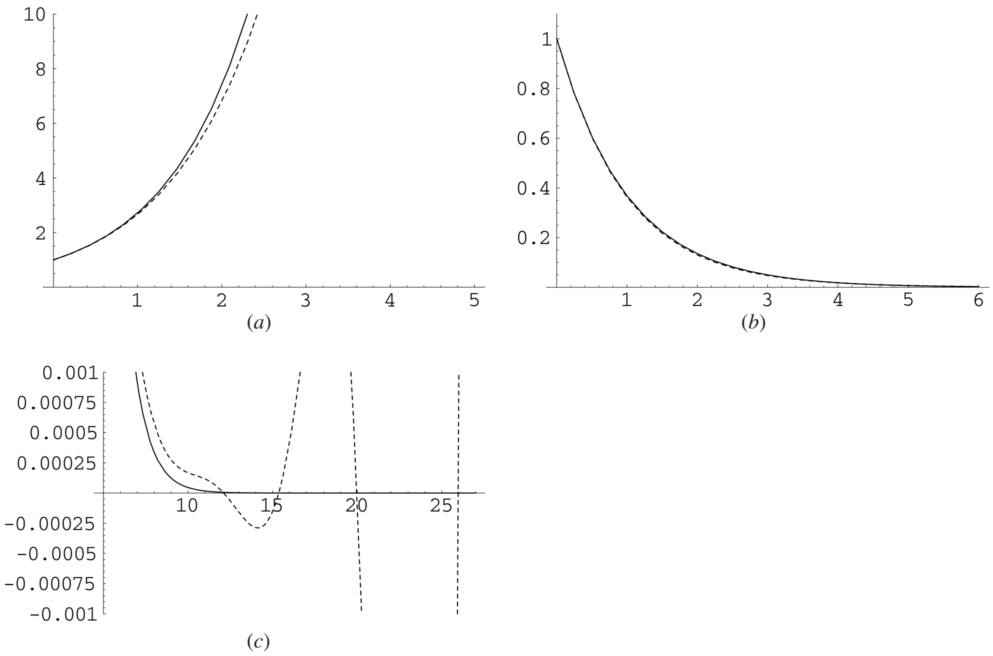


Figure 3. (*a*) Plot of the exponential function (continuous line) and symmetric q-exponential function (dashed line), for $\lambda = 1$ and $q = 1.3$. (*b*) Plot of the exponential function (continuous line) and symmetric q-exponential function (dashed line), for $\lambda = -1$ and $q = 1.3$. (*c*) Enlarged plot of the exponential function (continuous line) and symmetric q-exponential function (dashed line), for $\lambda = -1$ and $q = 1.3$.

exponential up to the first zero (for $\lambda < 0$) but it departs wildly oscillating beyond that point (see figure 3).

Comparing figures 1 and 3, we can see that the symmetric exponential function gives a better approximation than the right exponential. However, it can be easily shown that there is no initial condition for equation (4.8) such that the symmetric q-exponential vanishes for all subsequent points, as it was the case for the right exponential. So, there is no way to avoid the oscillations.

4.2. q-Gaussians

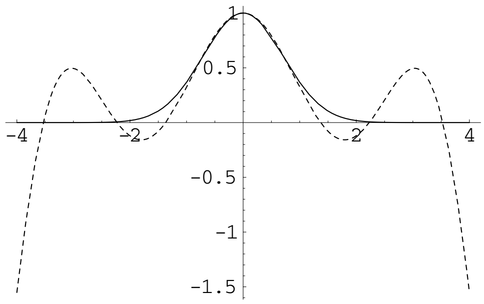
In this case equation (4.2), which has as a solution the Gaussian function, becomes the q-difference equation

$$\Delta G_{q,\lambda}(x) = -\lambda \beta_x \, x G_{q,\lambda}(x). \tag{4.10}$$

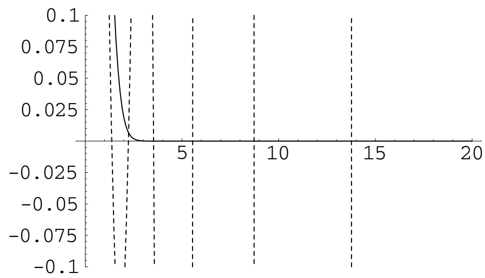
In the following we will discuss briefly the cases of the right and symmetric q-Gaussians solutions.

4.2.1. The right Gaussian. The difference equation (4.10) becomes a three-term recurrence equation

$$q^{-1} G^{+}_{q,\lambda}(q^2 x) - (q^{-1} + 1) G^{+}_{q,\lambda}(qx) + G^{+}_{q,\lambda}(x) = -\lambda (q-1)^2 x (x^2 \partial_x + 1) G^{+}_{q,\lambda}(x). \tag{4.11}$$



(*a*)



(*b*)

Figure 4. (*a*) Plot of the Gaussian function (continuous line) and right q-Gaussian function (dashed line), for $\lambda = 1$ and $q = 1.3$. (*b*) Enlarged plot of the Gaussian function (continuous line) and right q-Gaussian function (dashed line), for $\lambda = 1$ and $q = 1.3$.

Equation (4.11) is a differential-difference equation. So, it is quite difficult to find directly the solutions or even to discuss the general behaviour of them. We will assume, as shown in the case of the exponential function, that whenever the q-umbral series is convergent it will converge to the solution of the difference equation.

The q-umbral series supplies a solution to equation (4.11), given by

$$\widetilde{G}^{+}_{q,\lambda}(x) = \sum_{k=0}^{\infty} \frac{(-\lambda x^2)^k 2^k (2k-1)!(q-1)^{2k}}{\prod_{j=1}^{2k}(q^j - 1)}. \tag{4.12}$$

For $q > 1$ the series converges for all x, but for $q < 1$ the series diverges everywhere (for $x \neq 0$). Some plottings of $\widetilde{G}^{+}_{q,\lambda}$ are shown in figure 4. The behaviour is similar to that of the q-exponential, that is whenever we have a decreasing function of x, at a certain point it vanishes and beyond that point it starts to oscillate with increasing amplitudes, thus departing from the behaviour of the continuous Gaussian function.

4.2.2. The symmetric Gaussian. The recurrence relation of equation (4.10) for this case is an even more involved differential difference equation than equation (4.11), so we prefer not to write it down here. The q-umbral series solution is given by

$$\widetilde{G}^{s}_{q,\lambda}(x) = \sum_{k=0}^{\infty} \frac{(-\lambda x^2)^k (2k)!(q - 1/q)^{2k}}{k! \prod_{j=1}^{2k}(q^j - 1/q^j)}. \tag{4.13}$$

The radius of convergence is $R = \infty$. For the symmetric Gaussian function we have similar results as for the right Gaussian, which, however, are valid for any value of q (see figure 5).

In conclusion we can say that the decreasing asymptotic behaviour of the continuous functions is not fully reproduced by the corresponding umbral q-functions. The discrete functions approach the continuous ones up to the point where they vanish. Beyond this point they oscillate going far away from the continuous analogues. Therefore, a good parameter to measure the radius of the domain where the q-functions imitate the continuous functions is given by the first zero in the asymptotic region. This point is plotted in figure 6 for the q-exponentials and the q-Gaussians functions.

From figure 6 we see that the domain of convergence of the q-exponential to the exponential function increases in a monotonic continuous way as $q \to 1$ for the right

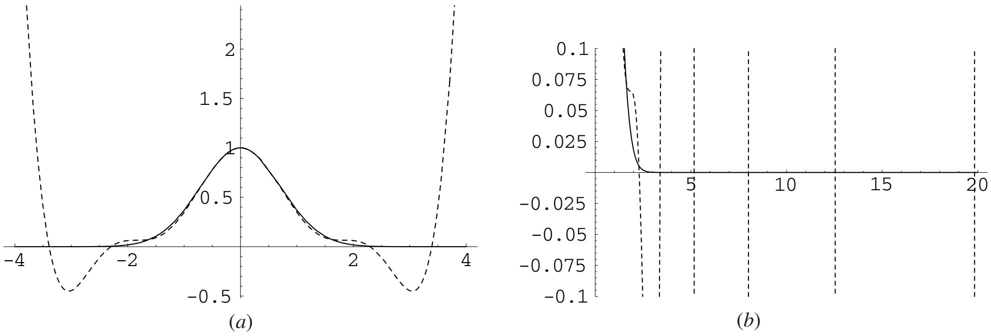


Figure 5. (a) Plot of the Gaussian function (continuous line) and symmetric q-Gaussian function (dashed line), for $\lambda = 1$ and $q = 1.3$. (b) Enlarged plot of the Gaussian function (continuous line) and symmetric q-Gaussian function (dashed line), for $\lambda = 1$ and $q = 1.3$.

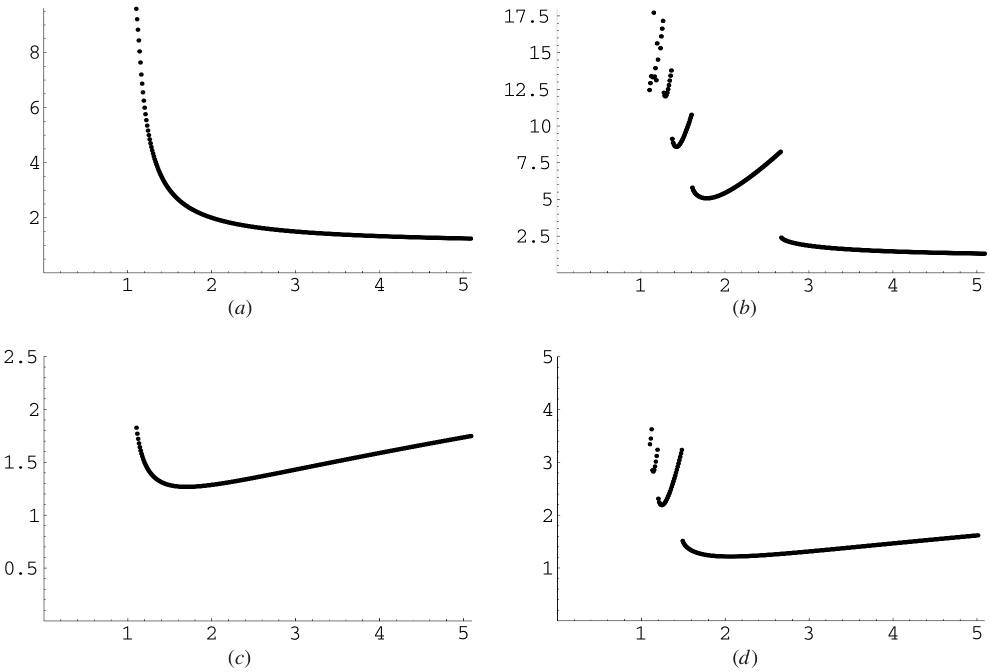


Figure 6. (a) Plot of the position of first zero of the right q-exponential as function of q for $\lambda = -1$. (b) Plot of the position of first zero of the symmetric q-exponential as function of q for $\lambda = -1$. (c) Plot of the position of first zero of the right q-Gaussian as function of q for $\lambda = 1$. (d) Plot of the position of first zero of the symmetric q-Gaussian as function of q for $\lambda = 1$.

exponential, while in the symmetric case there are discontinuities for small values of q. The situation is slightly different in the case of the Gaussian function. In this case there is a minimum value of the first zero, $q_0(\lambda)$, such that for $1 < q_0(\lambda) < q < \infty$ also the domain decreases. Below q_0 the domain increases as $q \to 1$. In the case of the symmetric Gaussian function we have again that for q small the function is discontinuous.

5. The discrete heat q-equation and its symmetries

Taking into account the q-umbral correspondence

$$\partial_x \to \Delta_x \qquad x \to \beta_x x \tag{5.1}$$

we obtain from any linear constant coefficient differential equation an operator equation which, when projected, gives us a q-discrete equation. In the case of the heat equation we get

$$\left(\partial_t - \partial_{xx}^2\right)u = 0 \quad \Longrightarrow \quad (\Delta_t - \Delta_{xx})u = 0. \tag{5.2}$$

Let us construct the symmetries for the q-discrete equation (5.2). We can apply the q-umbral correspondence also to the determining equations for the symmetry generators (2.6),

$$X_e \equiv Q\partial_u = (\tau\Delta_t + \xi\Delta_x u + fu)\partial_u$$

because they are linear in the coefficients ξ, τ and f. Since equation (5.2) is a second-order difference equation it is necessary to use the second prolongation. The corresponding determining equation is

$$\Delta_t^T Q - \Delta_{xx}^T Q\big|_{\Delta_{xx}u=\Delta_t u} = 0 \tag{5.3}$$

where Δ^T means a total derivative [12]. Equation (5.3) reads

$$\Delta_t(\xi\Delta_x u) + \Delta_t(\tau\Delta_t u) + \Delta_t(fu) - [\Delta_{xx}(\xi\Delta_x u) + \Delta_{xx}(\tau\Delta_t u) + \Delta_{xx}(fu)]|_{\Delta_{xx}u=\Delta_t u} = 0. \tag{5.4}$$

Making use of the Leibniz rule, from equation (5.4) we obtain the following set of equations (see also [16])

$$\begin{aligned} &D_x(\tau) = 0 && D_t(\tau) - 2D_x(\xi) = 0 \\ &D_t(\xi) - D_{xx}(\xi) - 2D_x(f) = 0 && D_t(f) - D_{xx}(f) = 0 \end{aligned} \tag{5.5}$$

where $D_x(\tau)1 = [\Delta_x, \tau]1 = \Delta_x\tau$. Moreover,

$$D_{xx}(f)1 = D_x(D_x(f))1 = [\Delta_x, [\Delta_x, f]]1 = \Delta_x\Delta_x f.$$

Note that these determining equations have formally the same expression for all the q-derivatives, for the continuous derivatives and also in the discrete case studied in [5]. From equation (5.1), the solution of this system (5.5) is

$$\begin{aligned} \tau &= \tau_2(\beta_t t)^2 + \tau_1(\beta_t t) + \tau_0 \\ \xi &= \tfrac{1}{2}(\tau_1 + 2\tau_2(\beta_t t))(\beta_x x) + \xi_1(\beta_t t) + \xi_0 \\ f &= \tfrac{1}{4}\tau_2(\beta_x x)^2 + \tfrac{1}{2}\tau_2(\beta_t t) + \tfrac{1}{2}\xi_1(\beta_x x) + \gamma \end{aligned} \tag{5.6}$$

where $\tau_0, \tau_1, \tau_2, \xi_0, \xi_1$ and γ are arbitrary functions of T_x, T_t and of q_x and q_t. By a suitable choice of these functions, we get the following representation of the symmetries

$$\begin{aligned} P_0^q &= (\Delta_t u)\partial_u \\ P_1^q &= (\Delta_x u)\partial_u \\ W^q &= u\partial_u \\ B^q &= (2(\beta_t t)\Delta_t u + (\beta_x x)\Delta_x u)\partial_u \\ D^q &= \left(2(\beta_t t)\Delta_t u + (\beta_x x)\Delta_x u + \tfrac{1}{2}u\right)\partial_u \\ K^q &= ((\beta_t t)^2\Delta_t u + (\beta_t t)(\beta_x x)\Delta_x u + \tfrac{1}{4}(\beta_x x)^2 u + \tfrac{1}{2}(\beta_t t)u)\partial_u \end{aligned} \tag{5.7}$$

that close into a six-dimensional Lie algebra, isomorphic to the symmetry algebra of the continuous heat equation. Another realization of this algebra was obtained in [17], by a different procedure and used to find symmetric solutions of the discrete heat equation.

Now we use the obtained symmetries to construct some solutions of the q-discrete equation (5.2).

Taking into account the symmetries P_0^q and P_1^q, choosing the q-parameters in such a way that $q_x = q_t = q$ and using the variable separation method [18], we can write the solution of the q-heat equation as

$$u(\beta_t t, \beta_x x) = v(\beta_t t) w(\beta_x x).$$

Then equation (5.2) reads

$$(\Delta_t v(\beta_t t)) w(\beta_x x) - v(\beta_t t)(\Delta_{xx}) w(\beta_x x) = 0. \tag{5.8}$$

From equation (5.8) we deduce that the functions v and w must satisfy the following equations:

$$\Delta_t v(\beta_t t) = \lambda v(\beta_t t) \qquad \Delta_x w(\beta_x x) = \sqrt{\lambda} w(\beta_x x). \tag{5.9}$$

Taking into account equation (4.3), we have

$$v(t) = v(\beta_t t)1 = \mathrm{e}^{\lambda \beta_t t} 1 = \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{[n]_q!} = e_q^{\lambda t}$$

and

$$w(x) = w(\beta_x x)1 = \mathrm{e}^{\sqrt{\lambda}\beta_x x} 1 = \sum_{n=0}^{\infty} \frac{\lambda^{n/2} x^n}{[n]_q!} = e_q^{\sqrt{\lambda} x}.$$

Hence, the solution will be

$$u(t, x) = e_q^{\lambda t} e_q^{\sqrt{\lambda} x}.$$

Let us consider now the symmetry reduction with respect to B^q [13]. In this case introducing the appropriate symmetry variable $\eta = \frac{\beta_x x}{\sqrt{\beta_t t}}$ we get

$$u(x, t) = \frac{u_0}{\sqrt{\beta_t t}} \exp\left[-\frac{(\beta_x x)^2}{4\beta_t t} \right] 1. \tag{5.10}$$

The solution (5.10) of the q-heat equation is meaningful as long as we are considering positives times and the value $t = 0$ is out of our time domain. In such a situation the solution (5.10) is entire and can be represented as a Taylor series and, thus, q-functions like the Gaussian or the square root are meaningful. The method would not provide a meaningful q-function if we would consider all values of t. However, in $t = 0$ also the boost solution of the continuous heat equation would be singular and, thus, meaningless.

6. Conclusions

In this paper we presented a q-extension of the umbral calculus and used it to provide solutions of linear q-difference and q-differential difference equations. In this way we obtained solutions which have the correct continuous limit.

The discretization procedure given by the recipe $\partial_x \longrightarrow \Delta_x$, $x \longrightarrow \beta_x x$ also works well for linear equations in the case of q-shifts operators. In particular, it preserves the classical Lie symmetries which are described by linear equations.

We study in detail the behaviour of the q-exponential and q-Gaussian functions and show their range of validity which depends on the q-discrete derivative operator under consideration. The domain of convergence of the q-function to the continuous function is characterized in terms of the zeros of the q-function. The results are usually better in the case of symmetric q-derivative operators.

Further work is in progress on the complete description of a coherent q-umbral calculus and a comparison of the discrete and q-discrete solutions.

Acknowledgments

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